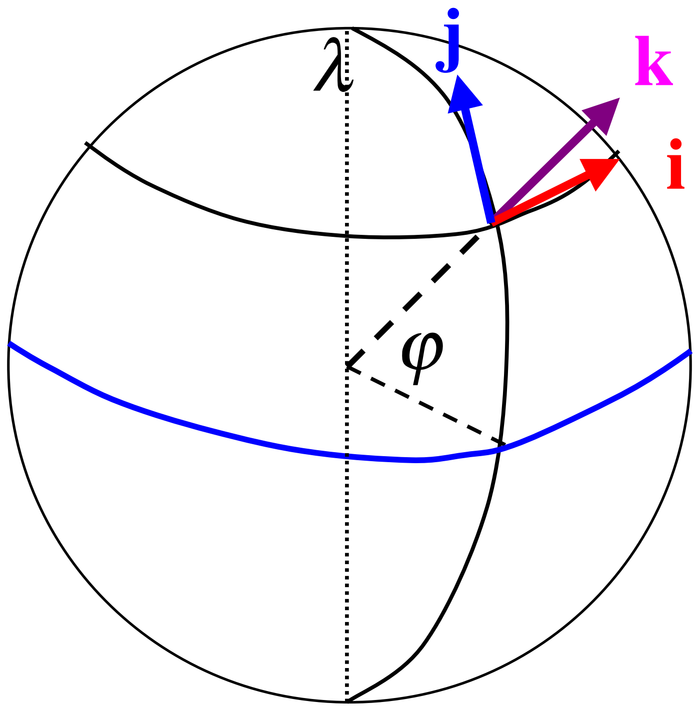
Chapter 2. The continuous equations



$$\mathbf{v}(\lambda,\varphi,z,t) = u(\lambda,\varphi,z,t)\mathbf{i} + v(\lambda,\varphi,z,t)\mathbf{j} + w(\lambda,\varphi,z,t)\mathbf{k}$$

$$\frac{d\mathbf{v}}{dt} = \frac{du}{dt}\mathbf{i} + u\frac{d\mathbf{i}}{dt} + \frac{dv}{dt}\mathbf{j} + v\frac{d\mathbf{j}}{dt} + \frac{dw}{dt}\mathbf{k} + w\frac{d\mathbf{k}}{dt}$$

2.2 Atmospheric equations of motion on spherical coordinates

Since the earth is nearly spherical, it is natural to use spherical coordinates.

Near the earth, gravity is almost constant, and the ellipticity of the earth is very small, so that one can accurately approximate scale factors by those appropriate for true spherical coordinates Phillips (1966, 1973, 1990).

The three velocity components are

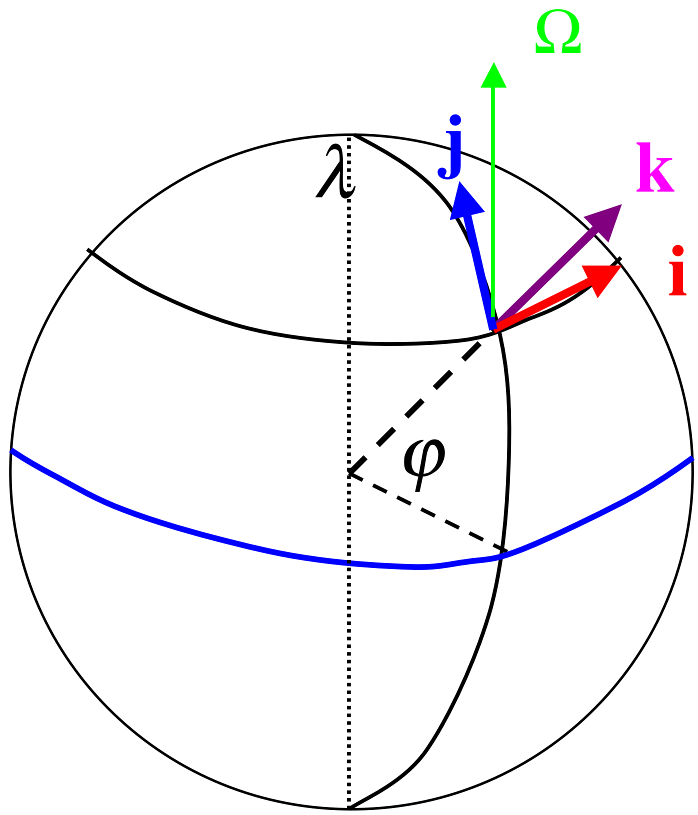
$$
\begin{aligned}
u &= \text{zonal (positive eastward)} = r \cos\varphi \frac{d\lambda}{dt} \\
v &= \text{meridional (positive northward)} = r \frac{d\varphi}{dt} \\
w &= \text{vertical (positive up)} = \frac{dr}{dt}
\end{aligned}
\tag{2.1}
$$

Note that $\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors in the three orthogonal spherical coordinates.

When the acceleration (total derivative of the velocity vector) is calculated, the rate of change of the unit vectors has to be included.

For example, geometrical considerations show that

$$\frac{d\mathbf{k}}{dt} = \frac{u}{r\cos\varphi}\frac{\partial \mathbf{k}}{\partial \lambda} + \frac{v}{r}\frac{\partial \mathbf{k}}{\partial \varphi} = \frac{u\mathbf{i}}{r\cos\varphi} + \frac{v\mathbf{j}}{r}.$$



$$\boxed{\mathbf{v}(\lambda,\varphi,z,t) = u(\lambda,\varphi,z,t)\mathbf{i} + v(\lambda,\varphi,z,t)\mathbf{j} + w(\lambda,\varphi,z,t)\mathbf{k}}$$

$$\boxed{\frac{d\mathbf{v}}{dt} = \frac{du}{dt}\mathbf{i} + u\frac{d\mathbf{i}}{dt} + \frac{dv}{dt}\mathbf{j} + v\frac{d\mathbf{j}}{dt} + \frac{dw}{dt}\mathbf{k} + w\frac{d\mathbf{k}}{dt}}$$

$$\boxed{\frac{d\mathbf{k}}{dt} = \frac{u}{r\cos\varphi}\frac{\partial\mathbf{k}}{\partial\lambda} + \frac{v}{r}\frac{\partial\mathbf{k}}{\partial\varphi} = \frac{u\mathbf{i}}{r\cos\varphi} + \frac{v\mathbf{j}}{r}}$$

When we include these time derivatives, take into account that $\mathbf{\Omega} = \Omega\sin\varphi\mathbf{k} + \Omega\cos\varphi\mathbf{j}$, and expand the momentum equation (2.1.19) into its three components, we obtain

$$\frac{du}{dt} = -\frac{\alpha}{r\cos\varphi}\frac{\partial p}{\partial\lambda} + F_\lambda + (2\Omega + \frac{u}{r\cos\varphi})(v\sin\varphi - w\cos\varphi)$$

$$\frac{dv}{dt} = -\frac{\alpha}{r}\frac{\partial p}{\partial\varphi} + F_\varphi - (2\Omega + \frac{u}{r\cos\varphi})u\sin\varphi - \frac{vw}{r}$$

$$\frac{dw}{dt} = -\alpha\frac{\partial p}{\partial r} - g + F_r + (2\Omega + \frac{u}{r\cos\varphi})u\cos\varphi + \frac{v^2}{r}$$

(2.2)

The terms proportional to $u / r\cos\varphi$ are known as "metric terms".

A "*traditional approximation*" (Phillips 1966) has been routinely made in numerical weather prediction, since most of the atmospheric mass is confined to a few tens of kilometers. This suggests that in considering the distance of a point to the center of the earth $r = a + z$, one can neglect *z* and replace *r* by the radius of the earth $a = 6371km$, replace $\partial / \partial r$ by $\partial / \partial z$,

and neglect the metric and Coriolis terms proportional to $\cos\varphi$.

Then the equations of motion in spherical coordinates become

$$\frac{du}{dt} = -\frac{\alpha}{a\cos\varphi}\frac{\partial p}{\partial\lambda} + F_\lambda + (2\Omega + \frac{u}{a\cos\varphi})v\sin\varphi$$

$$\frac{dv}{dt} = -\frac{\alpha}{a}\frac{\partial p}{\partial\varphi} + F_\varphi - (2\Omega + \frac{u}{a\cos\varphi})u\sin\varphi$$

$$\frac{dw}{dt} = -\alpha\frac{\partial p}{\partial z} - g + F_z$$

(2.3)

which possess the angular momentum conservation principle

$$\frac{d}{dt}[(u + \Omega a\cos\varphi)a\cos\varphi] = a\cos\varphi(-\frac{\alpha}{a\cos\varphi}\frac{\partial p}{\partial\lambda} + F_\lambda)$$

(2.4)

With the "traditional approximation" the total time derivative operator in spherical coordinates is given by

$$\frac{d(\,)}{dt} = \frac{\partial(\,)}{\partial t} + \frac{u}{a\cos\varphi}\frac{\partial(\,)}{\partial\lambda} + \frac{v}{a}\frac{\partial(\,)}{\partial\varphi} + w\frac{\partial(\,)}{\partial z} \quad (2.5)$$

and the 3-dimensional divergence that appears in the continuity equation by

$$\nabla_3 .\mathbf{v} = \frac{1}{a\cos\varphi}\left(\frac{\partial u}{\partial \lambda} + \frac{\partial v\cos\varphi}{\partial \varphi}\right) + \frac{\partial w}{\partial z} \qquad (2.6)$$