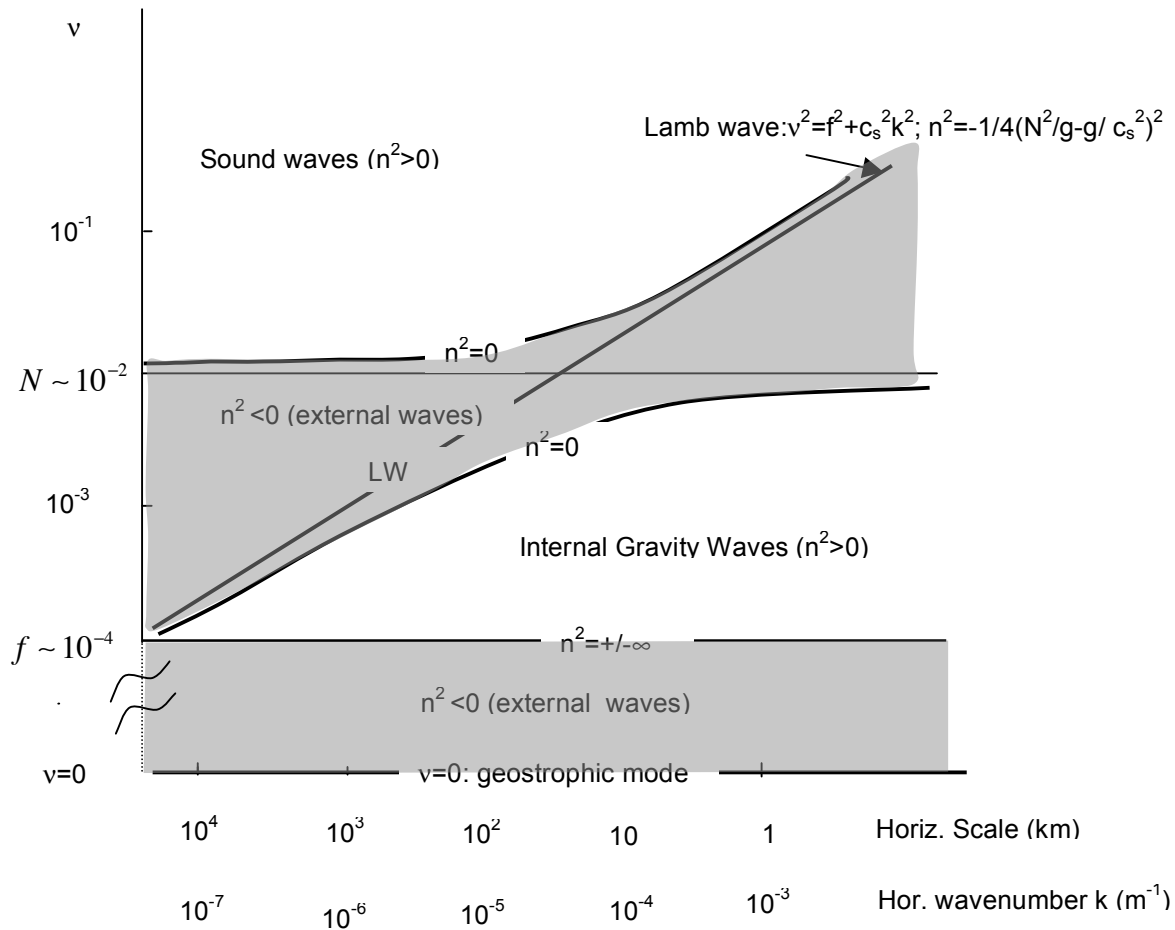


## Chapter 2. The continuous equations

$$n^2 = \frac{(\beta(v^2 - f^2) - c_s^2 k^2)(v^2 \alpha - N^2)}{c_s^2(v^2 - f^2)} - \frac{1}{4} \left( \beta \frac{N^2}{g} - \frac{g}{c_s^2} \right)^2$$

Frequency of small perturbations in an isothermal resting atmosphere.



## 2. 4 Filtering approximations

When we **neglect the time derivative** of one of the equations of motion, we convert it from a **prognostic** equation into a **diagnostic** equation, and **eliminate** one type of solution.

Physically, we eliminate a **restoring force** that supports a certain type of wave.

We call this a “filtering approximation”.

Use of the quasi-geostrophic filtering approximation that eliminates both sound and gravity waves made possible the successful forecast of Charney et al (1950).

Currently most global models and some regional models use the hydrostatic approximation, which filters sound waves. Eventually most models will avoid this approximation, not valid when horizontal scales are short.

In this section we explore the effect of the filtering approximations.

### 2.4.1 Quasi-geostrophic approximation:

With rotation, if we assume steady state solutions, and neglect all time derivatives ( $\nu=0$ ), we obtain from (3.19) the **geostrophic mode**, a **non-trivial** solution:

$$\begin{aligned} -ivU &= -ikP + fV && \text{a} \\ -ivV &= -fU && \text{b} \\ -iv\alpha W &= -Rg - \frac{dP}{dz} && \text{c} \\ -iv\beta R &= -ikU - \frac{dW}{dz} && \text{d (3.19)} \\ -iv\left(\frac{P}{c_s^2} - R\right) &= -W \frac{N^2}{g} && \text{e} \end{aligned}$$

$$V = ikP / f$$

$$U = 0$$

$$\frac{dP}{dz} = -Rg$$

$$\frac{dW}{dz} = 0 \quad \text{and}$$

$$P = c_s^2 R$$

For the continuous perturbation equations (3.17)

$$\begin{aligned}
 \frac{\partial u^*}{\partial t} &= +fv^* - \frac{\partial p'}{\partial x} & \text{a} \\
 \frac{\partial v^*}{\partial t} &= -fu^* & \text{b} \\
 \alpha \frac{\partial w^*}{\partial t} &= -\frac{\partial p'}{\partial z} - \rho'g & \text{c} \\
 \beta \frac{\partial \rho'}{\partial t} &= -\left(\frac{\partial u^*}{\partial x} + \frac{\partial w^*}{\partial z}\right) & \text{d} \\
 \frac{g}{C_p} \frac{\partial s^*}{\partial t} &= -w^*N^2 & \text{e} \\
 s^* &= C_p \left(\frac{p'}{c_s^2} - \rho'\right) & \text{f}
 \end{aligned}
 \tag{3.17}$$

this implies:

$$\begin{aligned}
 fv^* &= -\frac{\partial p'}{\partial x} & \text{(geostrophically balanced flow)} \\
 \frac{\partial v^*}{\partial t} &= 0 & \text{(steady state flow)} \\
 0 &= -\frac{\partial p'}{\partial z} - \rho'g & \text{(hydrostatically balanced flow)} & \quad (1.1) \\
 w^* &= 0 \\
 \frac{\partial w^*}{\partial z} &= \frac{\partial u^*}{\partial x} = 0 & \text{(horizontal, non-divergent flow)}
 \end{aligned}$$

and

$s^* = C_p \left( \frac{p'}{c_s^2} - \rho' \right) = 0$  (pressure perturbations are proportional to density perturbations multiplied by the speed of sound squared. This is true whenever the hydrostatic equation is valid).

This is the **“ultimate”** filtering approximation: it filters out sound waves, inertia and gravity oscillations.

For large horizontal scales (as the weather waves  $\sim 1000\text{km}$ ) we have to include the effects of varying vertical planetary rotation, and the f-plane becomes a  $\beta$ -plane:  
 $f = f_0 + \beta y$ .

When horizontal advection by the basic flow is included, the stationary geostrophic flow solution becomes quasi-stationary (slowly varying).

The waves corresponding to the geostrophic mode are Rossby-type waves with a frequency small compared with the Coriolis or inertial frequency  
 $\nu \approx Uk - \beta / k \sim 10^{-5} - 10^{-6} \text{ sec}^{-1}$ .

Rossby waves are quasi-geostrophic ( $\nu^2 \ll f^2$ ), hydrostatically balanced, and the flow is quasi-horizontal ( $w^* / H \ll U^* / L$ ), and therefore quasi-nondivergent ( $\nabla \cdot \mathbf{v}_h \approx 0$ ).

Note that this type of quasi-geostrophic solution, fundamental for NWP, is still present in the general

equations of motion, and survives as a solution when we make either the anelastic or the hydrostatic approximation in order to filter out sound waves.

#### 2.4.2 Quasi-Boussinesq or anelastic approximation (Ogura and Phillips, 1962):

We now make the marker  $\beta=0$  in (3.17)d.

$$\beta \frac{\partial \rho'}{\partial t} = -\left(\frac{\partial u^*}{\partial x} + \frac{\partial w^*}{\partial z}\right) \quad \text{d}$$

This means that we neglected the time derivative  $\frac{\partial \rho'}{\partial t}$  compared to the horizontal and vertical divergence components  $\nabla_H \cdot v^*, \frac{\partial w^*}{\partial z}$  in the continuity equation. In other words, the 3D mass divergence is much smaller than its horizontal and vertical components.

With this approximation, the equations become “anelastic”, i.e., they do not allow the presence of sound waves, which require 3D divergence and convergence for their propagation.

Consider the terms that are neglected in the FDR:

$$n^2 = \frac{(\beta(v^2 - f^2) - c_s^2 k^2)(v^2 \alpha - N^2)}{c_s^2(v^2 - f^2)} - \frac{1}{4} \left( \beta \frac{N^2}{g} - \frac{g}{c_s^2} \right)^2$$

1)  $(v^2 - f^2 \ll c_s^2 k^2)$ , i.e., the **frequency of retained solutions is much smaller than that of sound waves**, therefore this **filters out also the Lamb waves**, i.e., **horizontally propagating sound waves**.

2)  $(N^2 / g \ll g / c_s^2)$ . This approximation is justified if

$$\frac{N^2}{g} = \frac{1}{\theta_0} \frac{d\theta_0}{dz} \ll \frac{g}{\gamma R T_0}, \text{ i.e., } \frac{\gamma H}{\theta_0} \frac{d\theta_0}{dz} \ll 1.$$

In other words, the deep anelastic approximation is justified for a model for which the potential temperature does not change too much within the depth  $\gamma R T / g \sim 10 \text{ km}$ . This is a reasonable approximation for the standard troposphere (not for deep flow into the stratosphere), since for the troposphere:  $\Delta\theta_0 / \theta_0 \sim 30 \text{ K} / 300 \text{ K} \sim 0.1$ .

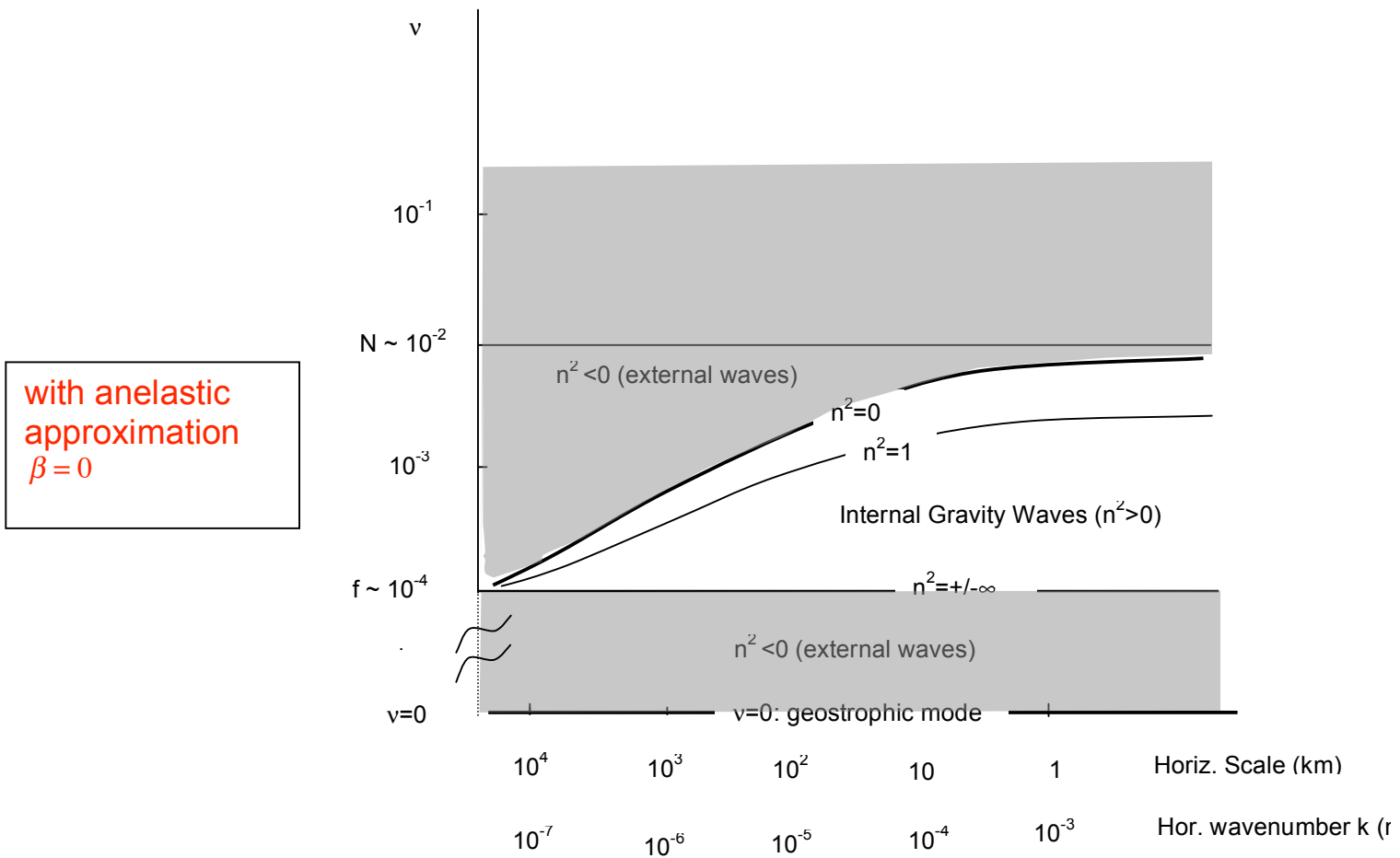
For models that are so shallow that not only  $\Delta\theta_0 / \theta_0 \ll 1$ , but also  $\Delta T_0 / T_0 \ll 1$ , we can also neglect  $\partial\rho_0 / \partial z$  in the continuity equation, and assume  $\nabla_3 \cdot \mathbf{v}' = 0$ , not just  $\nabla_3 \cdot \mathbf{v}^* = 0$ .

In this case we treat the atmosphere as if it was an incompressible fluid. This approximation is only accurate for very shallow atmospheric models (less than 1km depth), but very appropriate for ocean models, since water is well approximated as an incompressible fluid.

Fig. 2.4.1 shows schematically the FDR when we make the anelastic approximation. From (3.17), and letting  $\beta=0$ , we can derive the frequency of inertia-gravity waves with this approximation:

$$v^2 = f^2 \frac{n^2 + p^2}{n^2 + p^2 + k^2} + N^2 \frac{k^2}{n^2 + p^2 + k^2} \tag{1.2}$$

where  $p$  is like the inverse of a vertical wavelength  $p = \frac{1}{2} \frac{g}{c_s^2} \sim \frac{1}{20} \text{ km}^{-1}$ .





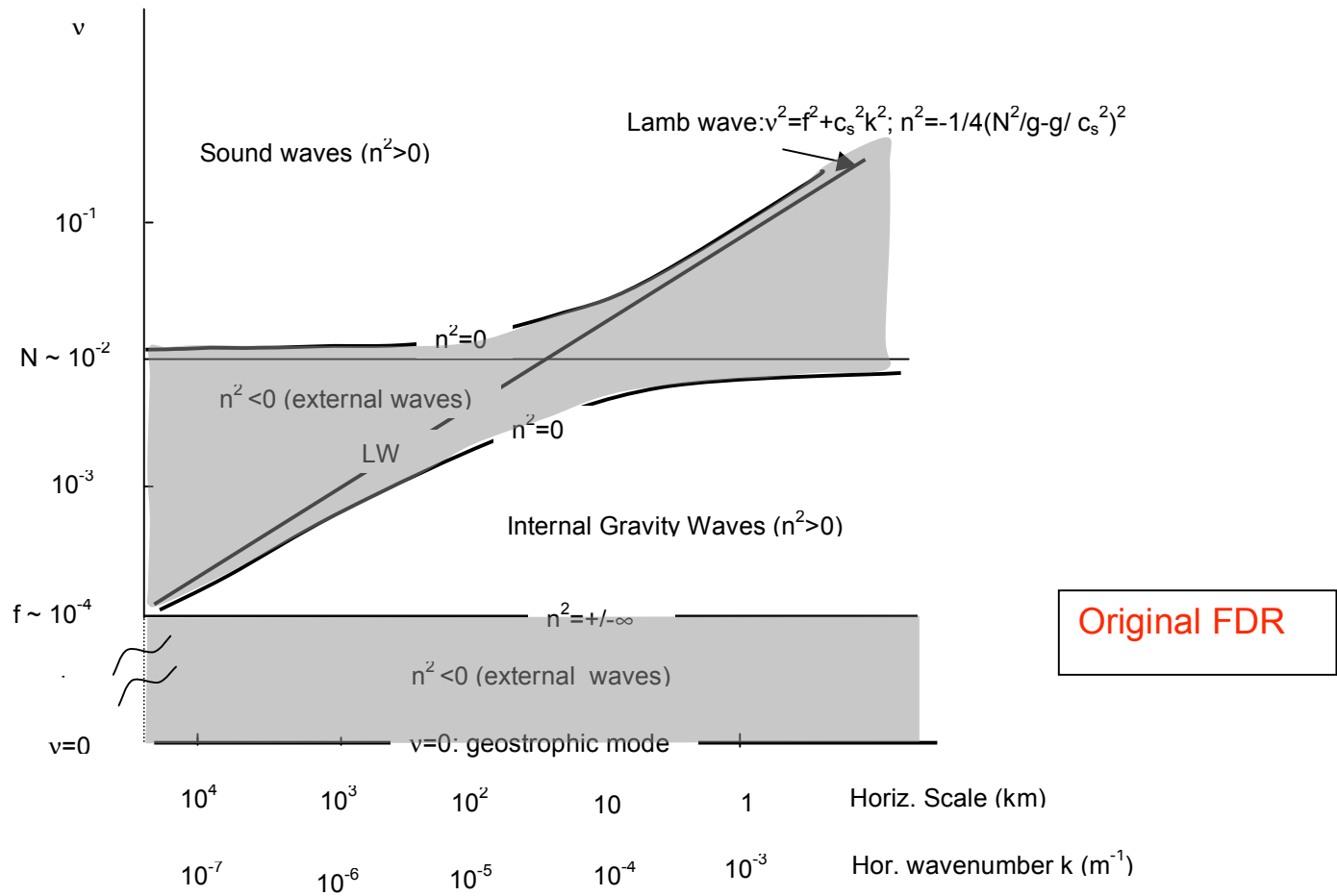


Fig. 2.4.1: Schematic of the frequencies of small perturbations in an isothermal resting atmosphere when the quasi-Boussinesq or anelastic approximation is made ( $\beta = 0$ ).

From (1.2) we see that, for internal ( $n^2 > 0$ ) inertia gravity waves,  $f^2 < v^2 < N^2$ , the frequency  $V$  is between the Coriolis and Brunt-Vaisala frequencies.

Note from Fig. 2.4.1 that for these waves,  $\partial v^2 / \partial k^2 > 0$ , but  $\partial v^2 / \partial n^2 < 0$ . This implies (since we can assume without loss of generality that  $k > 0$ ), that the horizontal group velocity for gravity waves  $\partial v / \partial k$  has the same sign as the phase velocity (gravity waves energy moves in the same direction as the phase speed in the horizontal). In the vertical the opposite is true: if the group velocity is upwards, which happens for example when gravity waves are generated by mountain forcing, the phase velocity is downwards.

Because the anelastic equation filters out acoustic internal waves (as well as the Lamb wave) it is widely used for problems in which the hydrostatic approximation cannot be made, as is the case for convection. For example, the ARPS model is based on deep anelastic equations.

### 2.4.3 Hydrostatic approximation

If we neglect the vertical acceleration  $\partial w^* / \partial t$  in the vertical momentum equation (3.17)c

$$\alpha \frac{\partial w^*}{\partial t} = -\frac{\partial p'}{\partial z} - \rho' g \quad \text{c}$$

, letting  $\alpha = 0$ , we get the FDR

$$n^2 = \frac{(v^2 - f^2 - c_s^2 k^2) N^2}{c_s^2 (v^2 - f^2)} - \frac{1}{4} \left( \frac{N^2}{g} - \frac{g}{c_s^2} \right)^2 \quad (1.3)$$

This FDR has two solutions, the **horizontally propagating external sound wave (Lamb wave)** solution, which unfortunately is retained:

$$v^2 = f^2 + c_s^2 k^2, \quad n^2 = -1/4 (N^2 / g - g / c_s^2)^2 \quad (1.4)$$

and inertia gravity waves (IGW).

From (1.3) we can derive for IGW's, using

$$N^2 = \kappa g / H, H = RT / g, c_s^2 = \gamma RT$$

$$n^2 = \frac{N^2 k^2}{v^2 - f^2} - \frac{1}{4H^2}, \text{ or } v^2 = f^2 + \frac{N^2 k^2}{n^2 + \frac{1}{4H^2}} \quad (1.5)$$

Exercise: derive (1.5) from (1.4).

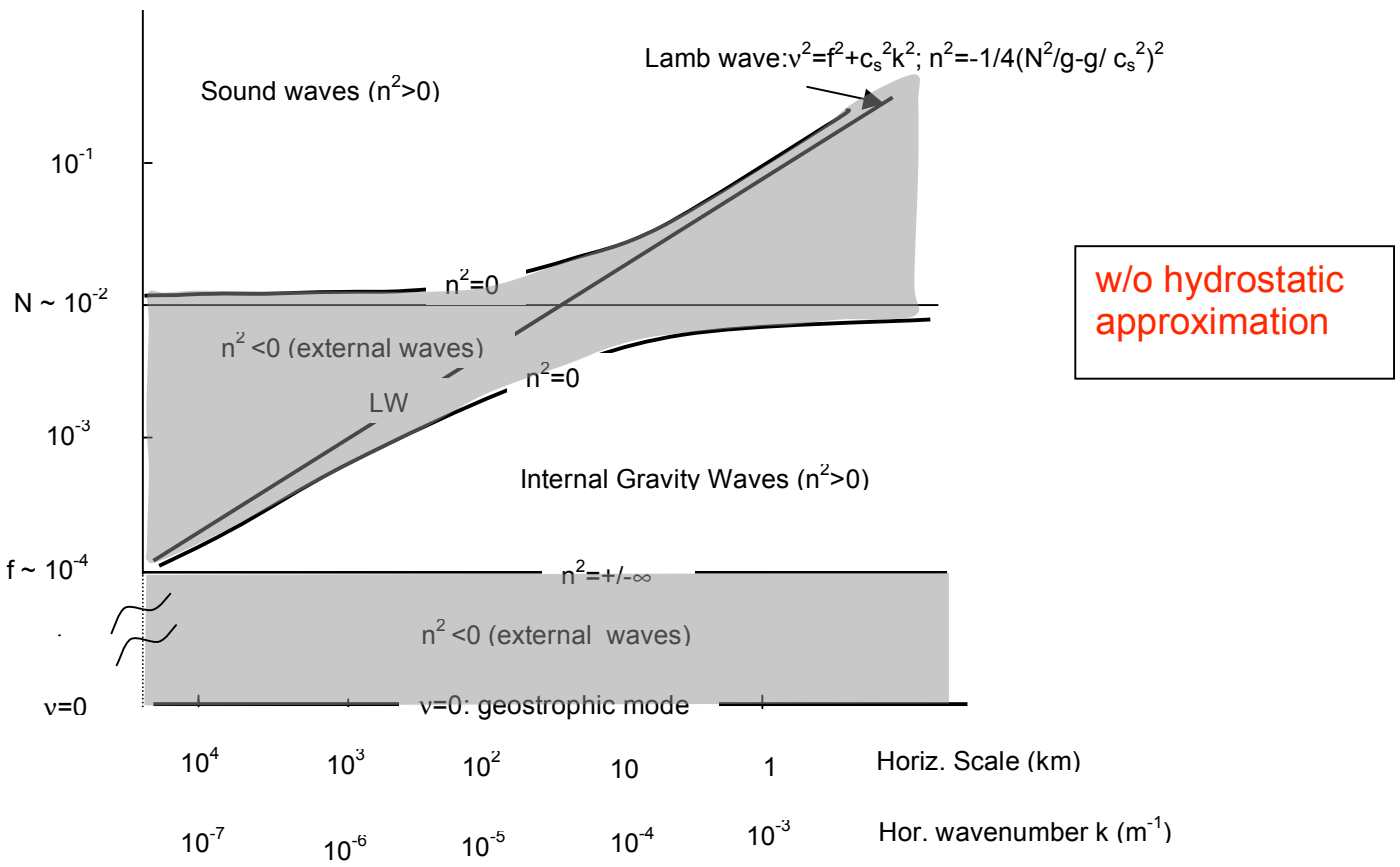


Fig. 2.4.2

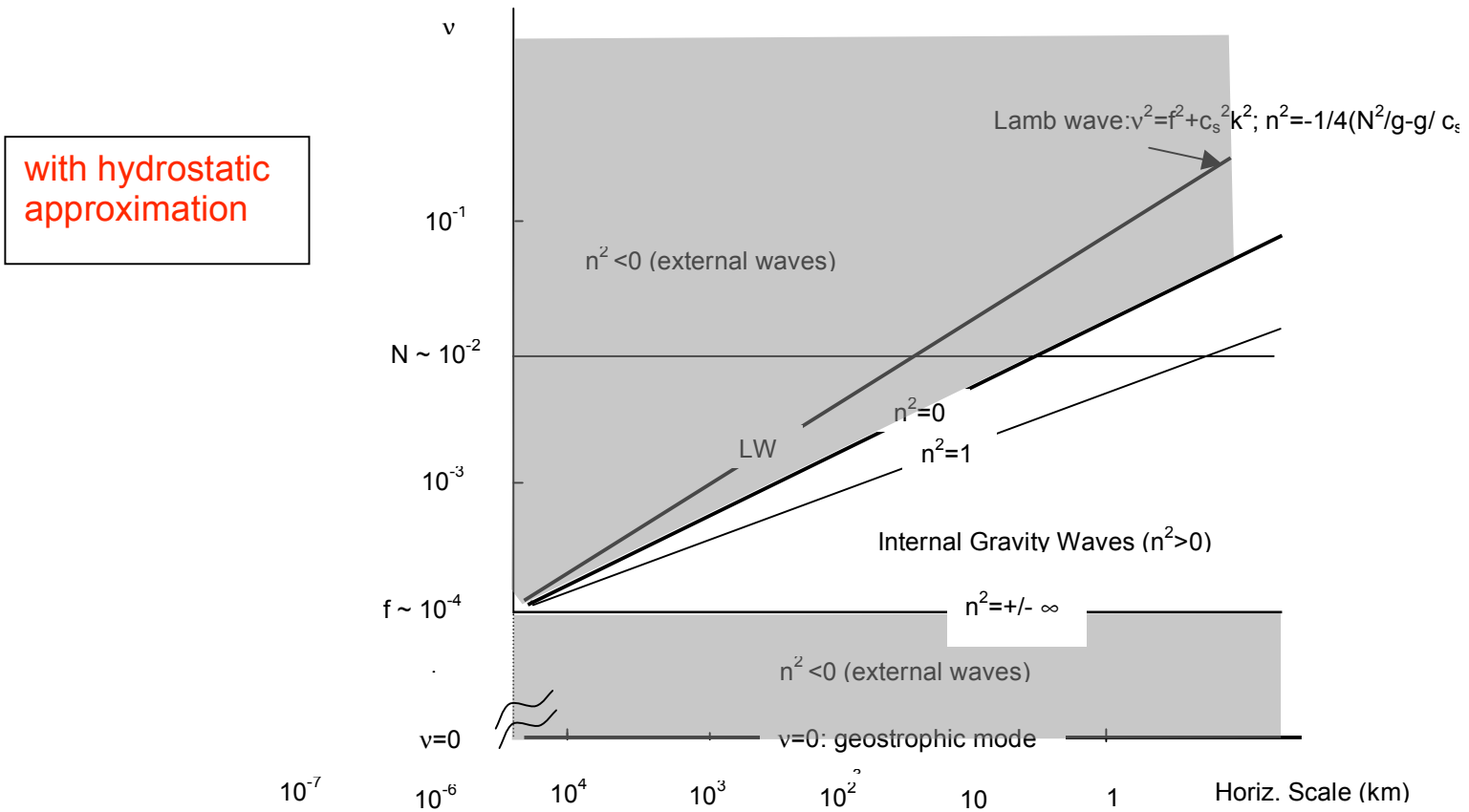


Fig. 2.4.2 shows the relationship between frequency, and horizontal and vertical wave numbers with the hydrostatic equation.

When are we justified in making the hydrostatic approximation? By taking  $\alpha=0$ , we neglected the **vertical acceleration** compared to  $\rho'/\rho_0g$ , the **buoyancy** within the fluid.

Note that it is not enough to find  $dw/dt \ll g$  to make the hydrostatic approximation: the vertical acceleration is small compared to gravity even for strong vertical motions, as in a cumulus cloud.

It can be shown by scale analysis that the hydrostatic approximation is valid as long as we are dealing with shallow flow ( $H/L \ll 1$ ) (exercise). For quasi-geostrophic flow, the condition for hydrostatic balance is valid even if  $H/L \sim 1$  (homework).

This implies that **the hydrostatic approximation is very accurate for models with grid sizes of the order of 100km or larger, and still quite acceptable for quasi-geostrophic flow, even when the horizontal grid size of the model approaches 10km.**

However, **the hydrostatic equation is not valid for models with grid sizes of the order of 10 km or less that attempt to resolve explicitly cumulus convection.** Fig. 2.4.2 shows that for high frequencies  $\nu \sim N$  or larger, or small horizontal scales the hydrostatic approximation distorts the original FDR (compare with Fig. 2.3.3).

We now summarize in Table 2.1 the characteristics of the different types of waves and the approximations that can be used to filter them out. (adapted from Zhang, pers. comm., 1996)

Type of wave (typical amplitude)	Phase speed	Restoring force	Filtering approximations
Acoustic (less than 0.1hPa, noise level)	$\sqrt{\gamma RT}$ (330m/sec)	Compression	Hydrostatic Anelastic Quasi-geostrophic
External gravity (if initial conditions are not balanced, 10hPa)	$\sqrt{gH}$ (320m/sec for H=10km)	Gravity	No free surface at the top or the bottom, or no net column mass convergence
Internal gravity (0.1-1hPa)	$\sim \frac{1}{k} \sqrt{\frac{N^2 k^2}{k^2 + n^2}} \sim N/k$ (50m/sec for L=30km)	Buoyancy (gravitational acceleration within fluid)	Neutral stratification (N=0), or $\frac{\partial \nabla \cdot \mathbf{v}_H}{\partial t} = 0$
Inertia	$f/k$ (15m/sec for L=1000km)	Coriolis force (f)	No rotation (f=0)
Rossby (20hPa)	$U - \beta/k^2$ (relative phase speed ~ 20-50m/sec depending on L)	Variation of f with latitude ( $\beta$ effect) $\frac{d\zeta}{dt} = -\beta v$	Constant f ( $\beta = 0$ )