

Chapter 2. The continuous equations

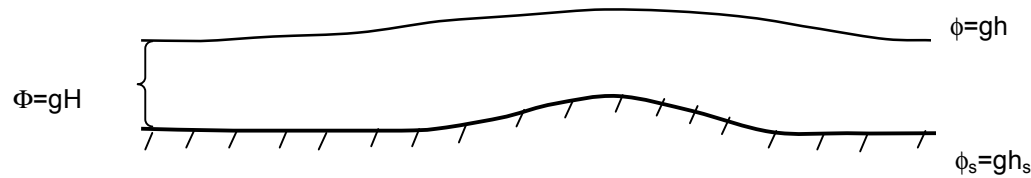


Fig. 2.5: Schematic of the shallow water model, a hydrostatic, incompressible fluid with a rigid bottom $h_s(x,y)$, a free surface $h(x,y,t)$, and horizontal scales L much larger than the mean vertical scale H .

2.5 Shallow water equations, quasigeostrophic filtering, and filtering of inertia-gravity waves

The shallow water equations (SWE, fig. 2.5), are valid for an **incompressible hydrostatic** motion of a fluid with a free surface $h(x,y,t)$.

“Shallow” means that the vertical depth is much smaller than the typical horizontal depth, which justifies the hydrostatic approximation.

These equations are appropriate for representing a shallow mass of water (e.g., shallow sea, river flow, storm surges).

They are also **a prototype of the primitive equations based on the hydrostatic approximation** and are frequently used to test numerical schemes.

The SWE horizontal momentum equations are

$$\frac{d\mathbf{v}}{dt} = -f\mathbf{k}\times\mathbf{v} - \nabla\phi \quad (5.1)$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$, $\mathbf{v} = \mathbf{v}_H = u\mathbf{i} + v\mathbf{j}$, and $\phi = gh$.

The continuity equation is

$$\frac{d(\phi - \phi_s)}{dt} = -(\phi - \phi_s)\nabla\cdot\mathbf{v} , \quad (5.2a)$$

which can also be written as

$$\frac{\partial\phi}{\partial t} = \nabla\cdot[(\phi - \phi_s)\mathbf{v}] . \quad (5.2b)$$

Here $\phi_s = gh_s(x, y)$ and h_s is the bottom topography.

Exercise 2.5.1: derive the SWE from the Primitive Equations (PE) assuming: hydrostatic, incompressible motion, and that the horizontal velocity is uniform in height. Is the vertical velocity uniform in height as well?

We derive the equation of **conservation of potential vorticity**: expanding the total derivative of the momentum equation, and making use of the useful relationship

$$\mathbf{v}_H \cdot \nabla \mathbf{v}_H = \nabla(\mathbf{v}_H^2 / 2) + \zeta \mathbf{k} \times \mathbf{v}_H \quad \text{where } \zeta = \mathbf{k} \cdot \nabla \times \mathbf{v}_H$$

we obtain

$$\frac{\partial \zeta}{\partial t} + \mathbf{v} \cdot \nabla \zeta + \zeta \nabla \cdot \mathbf{v} = -f \nabla \cdot \mathbf{v} - \mathbf{v} \cdot \nabla f \quad (5.2)$$

or (since $\frac{df}{dt} = \mathbf{v} \cdot \nabla f$)

$$\frac{d(f + \zeta)}{dt} = -(f + \zeta) \nabla \cdot \mathbf{v} \quad (5.3)$$

which indicates that the absolute vorticity of a parcel of “water” increases proportionally to its convergence (or vertical stretching). Eliminating the divergence from (5.4) and (5.2a), we obtain

$$\frac{d}{dt} \left(\frac{f + \zeta}{\phi - \phi_s} \right) = 0 \quad (5.4)$$

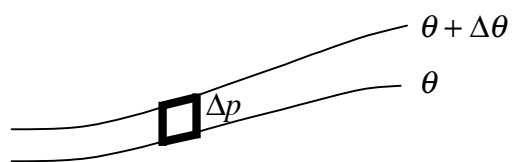
The potential vorticity is the absolute vorticity divided by the depth of the fluid:

$$q = \left(\frac{f + \zeta}{\phi - \phi_s} \right) \quad (5.5)$$

A parcel moving around conserves its potential vorticity!

The conservation of potential vorticity is an **extremely powerful dynamical constraint**.

In a multilevel primitive equations model, the **isentropic potential vorticity** (absolute vorticity divided by the distance between two surfaces of constant potential temperature) is also individually conserved:



$$\frac{d}{dt} \left(\frac{f + \zeta}{\Delta p / \Delta \theta} \right) = 0$$

Or, as the size of the parcel goes to zero, $\frac{d}{dt} \left((f + \zeta) \frac{\partial \theta}{\partial p} \right) = 0$

If the initial potential vorticity distribution is accurately represented in a numerical model, and the model is able to transport potential vorticity accurately, then the forecast will also be accurate.

Note that IPV can change with time if there is heating or friction because heating changes θ and friction changes the vorticity.

Frequency Dispersion Relationship in SWE:

We now consider small perturbations on a flat bottom and a mean height $\Phi = gH = \text{const.}$, on a constant f-plane.

$$\frac{\partial \mathbf{v}'}{\partial t} = -f \mathbf{k} \times \mathbf{v}' - \nabla \phi' \quad (5.6)$$

and

$$\frac{\partial \phi'}{\partial t} = -\Phi \nabla \cdot \mathbf{v}' \quad (5.7)$$

(note that (5.6) and (5.7) are the same equations as in the case 5 of horizontal sound (Lamb) waves, with

$$gH = c_s^2 = \gamma RT_o, \quad g\phi = \frac{p'}{\rho_0})$$

Assume solutions of the form $(u', v', \phi') e^{i(kx - vt)}$ and get

$$v(v^2 - f^2 - \Phi k^2) = 0 \quad (5.8)$$

with three solutions for v:

$$v^2 = f^2 + \Phi k^2 \quad (5.9)$$

the frequency of **inertia-gravity waves**, analogous to the inertia-Lamb wave, and $v=0$, the **geostrophic mode**.

As we saw with the primitive equations, this is a geostrophic, non-divergent steady-state solution

$$\frac{\partial}{\partial t} = 0, \quad \mathbf{v} = \frac{\mathbf{k} \times \nabla \phi}{f}, \quad \nabla \cdot \mathbf{v} = 0.$$

Following Arakawa (1997), we compare the FDR of IGW in the SWE with the FDR of a 3-dimensional isothermal equation using the hydrostatic approximation:

$$v^2 = f^2 + c_s^2 k^2 \text{ for Lamb waves, and}$$

$$v^2 = f^2 + \frac{N^2 k^2}{n^2 + \frac{1}{4H^2}} \text{ for inertia-gravity waves.}$$

We see that the SWE FDR $v^2 = f^2 + \Phi k^2$ is analogous to internal inertia-gravity waves for an isothermal hydrostatic atmosphere (eq.3.26) if we define an equivalent depth such that $\Phi = gh_{eq}$:

$$h_{eq} = \frac{N^2 / g}{n^2 + \frac{1}{4H_0^2}} = \frac{d \ln \theta_0 / dz}{n^2 + \frac{1}{4H_0^2}} \quad (5.10)$$

and is analogous to the (external) inertia Lamb waves if we define the equivalent depth as

$$h_{eq} = c_s^2 / g = \gamma H_0 \quad (5.11)$$

So, the SWE can be used as a toy model of the primitive equations with the inertia-Lamb wave and inertial-internal gravity waves by choosing the equivalent depth $\Phi = gh_{eq}$ appropriately.

2.5.1 Quasigeostrophic scaling for the SWE

If we want to filter the inertia-gravity waves (IGW), as Charney did in the first successful numerical weather forecasting experiment (Chapter 1), we can develop a quasi-geostrophic version of the SWE.

Do it first on an f-plane $f = f_0$

Assume that the atmosphere is in *quasi-geostrophic* balance:

$$\mathbf{v} = \mathbf{v}_g + \mathbf{v}_{ag} = \mathbf{v}_g + \varepsilon \mathbf{v}'$$

where we assume that the typical size of the ageostrophic wind is much smaller (order $\varepsilon = U / fL$, the Rossby number) than the geostrophic wind $\varepsilon \mathbf{v}' \ll \mathbf{v}_g$, and that the same is true for their time derivatives $\varepsilon \frac{\partial \mathbf{v}'}{\partial t} \ll \frac{\partial \mathbf{v}_g}{\partial t}$.

The geostrophic wind is given by

$$\mathbf{v}_g = \frac{1}{f} \mathbf{k} \times \nabla \phi . \quad (5.12)$$

Plugging these into the perturbation equations (5.6) and (5.7) we obtain

$$\frac{\partial \mathbf{v}_g}{\partial t} + \varepsilon \frac{\partial \mathbf{v}'}{\partial t} = -\nabla \phi - f \mathbf{k} \times \mathbf{v}_g - \varepsilon f \mathbf{k} \times \mathbf{v}' = -\varepsilon f \mathbf{k} \times \mathbf{v}' \quad (5.13)$$

In this equation, the dominant terms (pressure gradient and Coriolis force on the geostrophic flow) cancel each other (geostrophic balance), so that the smaller effect of the Coriolis force acting on the ageostrophic flow is left to balance the time derivative.

$$\frac{\partial \phi}{\partial t} = -\Phi \nabla \cdot \mathbf{v}_g - \varepsilon \Phi \nabla \cdot \mathbf{v}' = -\varepsilon \Phi \nabla \cdot \mathbf{v}' \quad (5.14)$$

Here the geostrophic wind is non-divergent, so that the time derivative of the pressure is given by the divergence of the smaller ageostrophic wind.

From (5.13) and (5.14) we can conclude that

$$\frac{\partial \mathbf{v}_g}{\partial t} \quad \text{and} \quad \frac{\partial \phi}{\partial t} \quad \text{are of order } \varepsilon$$

i.e., the geostrophic flow changes *slowly* (it is almost stationary compared with other types of motion),

and that $\frac{\partial \mathbf{v}_{ag}}{\partial t} = \varepsilon \frac{\partial \mathbf{v}'}{\partial t}$, which is smaller than $\frac{\partial \mathbf{v}_g}{\partial t}$, is of order ε^2 .

With *quasi-geostrophic scaling* we neglect terms of $O(\varepsilon^2)$ and we obtain the linearized QG SWE:

$$\begin{aligned}
 \frac{\partial \mathbf{v}_g}{\partial t} &= -\nabla \phi - f \mathbf{k} \times \mathbf{v} = -f \mathbf{k} \times \mathbf{v}_{ag} & a \\
 \frac{\partial \phi}{\partial t} &= -\Phi \nabla \cdot \mathbf{v} = -\Phi \nabla \cdot \mathbf{v}_{ag} & b \\
 \mathbf{v}_g &= 1/f \mathbf{k} \times \nabla \phi & c
 \end{aligned}
 \tag{5.15}$$

Note that in (5.15) there is only one independent time derivative because of the geostrophic relationship (we lost the other two time derivatives when we neglected the term $\frac{\partial \mathbf{v}_{ag}}{\partial t}$).

Physically, this means that we only allow divergent motion to exist as required to maintain the quasi-geostrophic balance, and eliminate the degrees of freedom necessary for the propagation of gravity waves.

We can rewrite (5.15) as

$$\begin{aligned}
\frac{\partial u_g}{\partial t} &= -\frac{\partial \phi}{\partial x} + fv = fv_{ag} & a \\
\frac{\partial v_g}{\partial t} &= -\frac{\partial \phi}{\partial y} + fu = -fu_{ag} & b \\
\frac{\partial \phi}{\partial t} &= -\Phi \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = \Phi \left(\frac{\partial u_{ag}}{\partial x} + \frac{\partial v_{ag}}{\partial y} \right) & c \quad (5.16) \\
u_g &= -\frac{1}{f} \frac{\partial \phi}{\partial y}; v_g = \frac{1}{f} \frac{\partial \phi}{\partial x} & d
\end{aligned}$$

We can compute the equation for the geostrophic vorticity evolution from (5.16) by taking the x-derivative of b minus the y-derivative of a:

$$\frac{\partial \zeta}{\partial t} = -f_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \beta v \quad (5.17)$$

where the last term in (5.17) appears if we are on a beta-plane: $f = f_0 + \beta y$.

Then we can eliminate the (ageostrophic) divergence between (5.17) and (5.16)c and obtain the linear quasi-geostrophic potential vorticity equation on a beta-plane:

$$\frac{\partial}{\partial t} \left(\frac{\zeta}{f_0} - \frac{\phi}{\Phi} \right) = -\frac{\beta}{f_0^2} \frac{\partial \phi}{\partial x} \quad (5.18)$$

or, since $\zeta = \frac{\nabla^2 \phi}{f_0}$,

$$\frac{\partial}{\partial t} \left(\frac{\nabla^2 \phi}{f_0^2} - \frac{\phi}{\Phi} \right) = - \frac{\beta}{f_0^2} \frac{\partial \phi}{\partial x} \quad (5.19)$$

Note that there is a single independent variable (ϕ) so that there is a single solution for the frequency. If we neglect the β -term (i.e., assume an f-plane) and allow for plane wave-type solutions $\phi = F e^{i(kx-vt)}$, the only solution of the FDR in (5.19) is $v=0$, the geostrophic mode. This confirms that by eliminating *the time derivative* of the ageostrophic (divergent) wind \mathbf{V}_{ag} , we have eliminated the inertia-gravity wave solution.

If we assume a beta-plane, i.e. keep the β term in (5.19), the quasi-geostrophic FDR becomes

$$v = \frac{-\beta k}{k^2 + f_0^2 / \Phi} \quad (5.20)$$

These are Rossby waves, the essential “weather waves”.

As shown in table 2.1, have rather large amplitudes (up to 50 hPa). The ageostrophic flow associated with these waves is responsible for the upward motion that produces precipitation ahead of the troughs.

In a multilevel model, the FDR (5.20) can be used with the equivalent depths (5.10), (5.11) applied to the baroclinic (internal) and barotropic Rossby waves respectively.

With these definitions, we can say that the waves in the atmosphere are analogous to the SWE waves.

However because the h_{eq} appears as a separation constant in the definition of the normal modes of the atmosphere, the equivalent depth depends on the vertical wavenumber, and on the type of wave considered (Lamb or IGW).

The QG Potential Vorticity equation (PVE) for nonlinear

$$\text{SWE is } \left(\frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y} \right) \left(\frac{\nabla^2 \phi}{f_0^2} - \frac{\phi}{\Phi} \right) = - \frac{\beta}{f_0^2} \frac{\partial \phi}{\partial x} \quad (5.21)$$

using similar scaling arguments.

If we add a basic flow $u_{g(total)} = U + u_g$; $v_{g(total)} = v_g$, it becomes

$$\left(\frac{\partial}{\partial t} + (U + u_g) \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y} \right) \left(\frac{\nabla^2 \phi}{f_0^2} - \frac{\phi}{\Phi} \right) = - \frac{\beta}{f_0^2} \frac{\partial \phi}{\partial x}$$

2.5.2 Inertia-gravity waves in the presence of a basic flow

As we just saw, the SWE are a simple version of the primitive equations, which are widely used to understand numerical and dynamical processes in primitive equations. As we noted in Chapter 1, filtered quasi-geostrophic models have been substituted by PE models for NWP, because the quasi-geostrophic filtering is not an accurate approximation (it assumes that the Rossby number is much smaller than 1). Recall that quasi-geostrophic filtering was introduced by Charney et al (1950) in order to eliminate the problem of gravity waves, which requires a small time step, and whose high frequencies produced a huge time derivative in

Richardson's computation, masking the time derivative of the actual weather signal.

An alternative way to deal with the presence of fast gravity waves without resorting to quasi-geostrophic filtering is the use of **semi-implicit time schemes** (to be discussed in Chapter 3).

Consider small perturbations in the SWE including a basic flow U in the x -direction. Then the total linearized time

derivative becomes $\frac{d}{dt} = \frac{\partial}{\partial t} + U \frac{\partial}{\partial x}$

In that case, when we assume solutions of the form

$Ae^{i(kx-vt)}$, $\frac{d}{dt} = i(-v + kU)$. Therefore the FDR remains the

same except that V is replaced by $v - kU$.

The FDR for small perturbations in the SWE with a basic flow U is therefore

$$(v - kU)[(v - kU)^2 - f^2 - \Phi k^2] = 0 \quad (5.22)$$

As noted before, it has three solutions, the quasi-geostrophic flow (which is steady state, except for the uniform translation with speed U), and two solutions for the inertia-gravity waves, modified by the basic flow translation:

$$(v_G - kU) = 0 \text{ (Geostrophic mode) and}$$

$$[(v_{IGW} - kU)^2 - f^2 - \Phi k^2] = 0 \text{ (inertia gravity waves, modified by the basic flow } U)$$

The phase speed of the IGW is given by

$$c_{IGW} = \frac{v_{IGW}}{k} = U \pm \sqrt{\frac{f^2}{k^2} + \Phi} \quad (5.23)$$

Finally, we note that for the Lamb wave (as well as for the external GW), the phase speed of the IGW is dominated by the term $\sqrt{\Phi} \approx \sqrt{g \cdot 10km} \approx 300m/sec$. As we will see in section 3.2.5, it is possible to avoid using costly small time steps by means of a **semi-implicit time scheme**.

An implicit time scheme has no constraint on the time step. Therefore, in a semi-implicit scheme, the terms that give rise to the fast gravity waves, namely the horizontal divergence and the horizontal pressure gradient are written implicitly, while the rest of the SWE terms can be written explicitly.

The terms generating the GW are underlined in the nonlinear SWE:

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{\partial \phi}{\partial x} + fv \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= -\frac{\partial \phi}{\partial y} - fu \\ \frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} &= -\Phi \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] - (\phi - \Phi) \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] \end{aligned} \quad (5.24)$$