Review of Probability

Wilks, Chapter 2

AOSC630
Spring, 2008

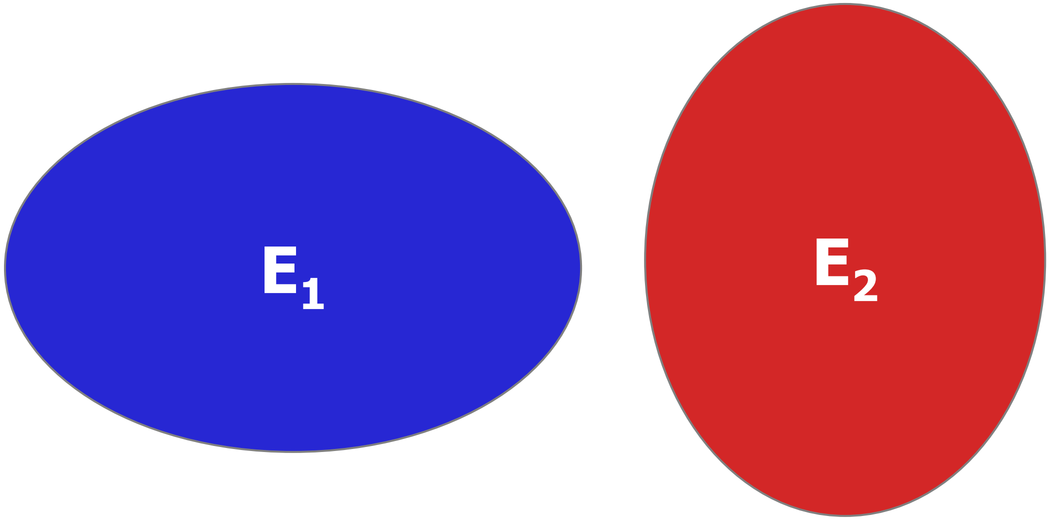
Definition

- **Event**: set, class, or group of possible uncertain outcomes
  - **A compound event** can be decomposed into two or more (sub) events
  - **An elementary event** cannot
- **Sample space (event space), S**
  - The set of all possible elementary events
- **MECE (Mutually Exclusive & Collectively Exhaustive)**
  - **Mutually Exclusive**: no more than one of the events can occur
  - **Collectively Exhaustive**: at least one of the events will occur

  ➔ ***A set of MECE events completely fills a sample space***

Probability Axioms

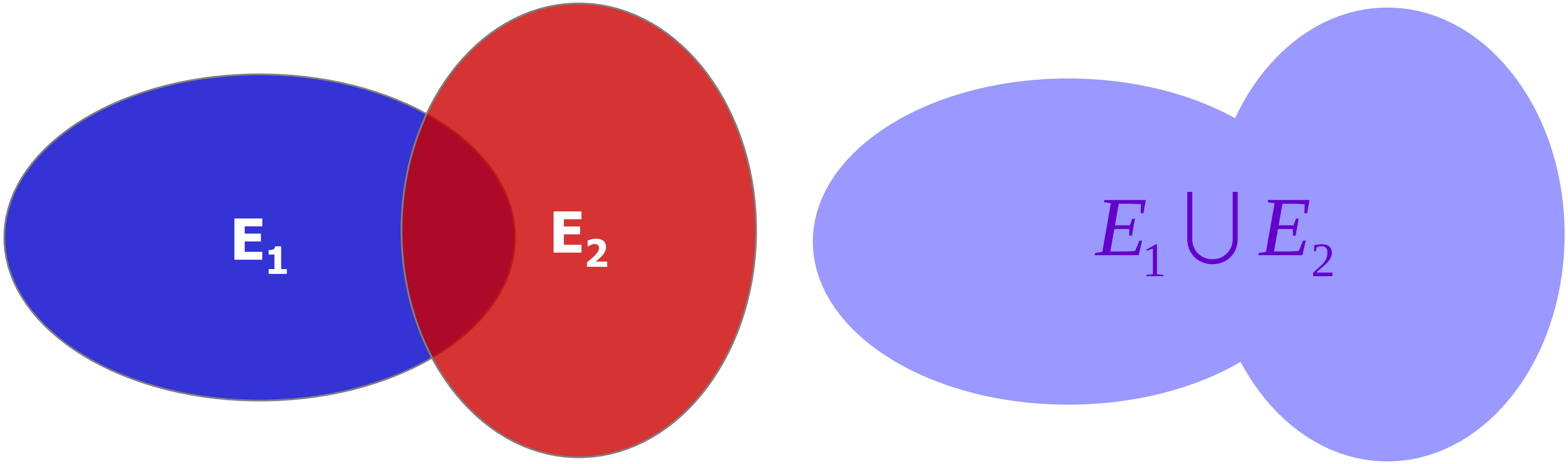
- $P(A) \geq 0$
- $P(S) = 1$
- If $(E_1 \cap E_2) = 0$, i.e., if E_1 and E_2 exclusive, then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$



Probability

- **Probability ~ Frequency**
  - P(E) = $\lim_{n\to\infty} \frac{\#E = yes}{total\_n}$
- **If** $E_2 \subseteq E_1$**, then** $P(E_1) \geq P(E_2)$
- $P(E_1 \bigcup E_2) = P(E_1) + P(E_2) - P(E_1 \bigcap E_2)$

Venn Diagrams



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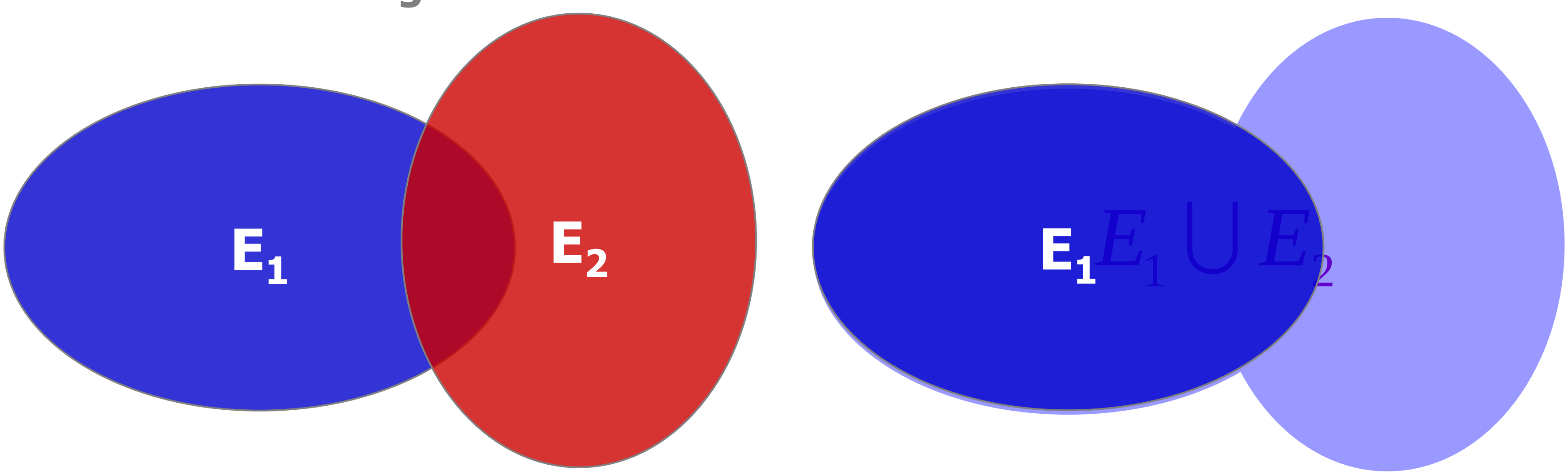

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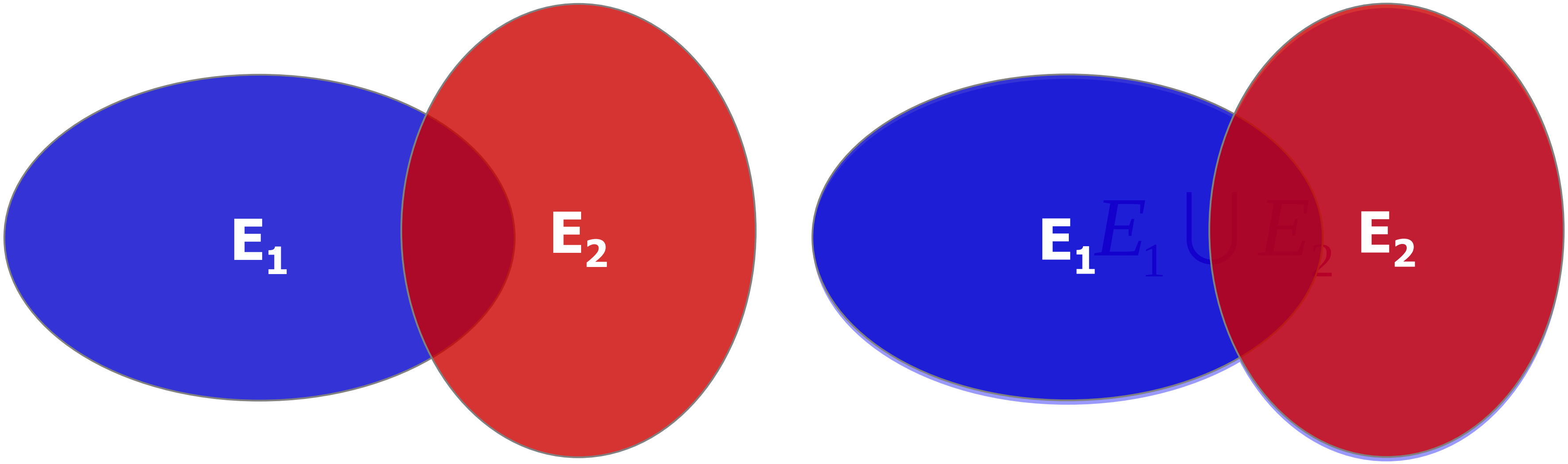

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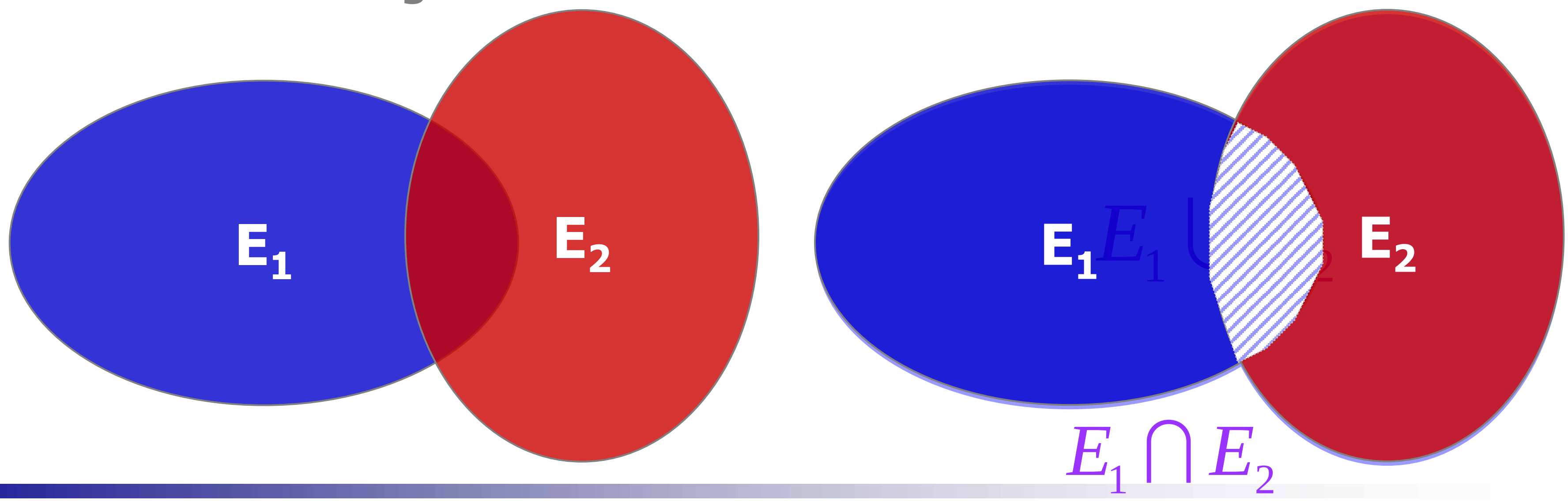

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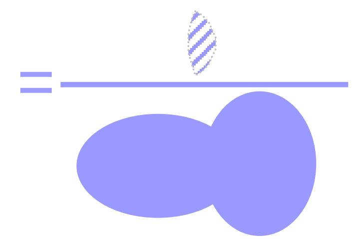
Recall Threat Score (TS)

$$= \frac{P(F = yes \bigcap Ob = yes)}{P(F = yes \bigcup Ob = yes)}$$



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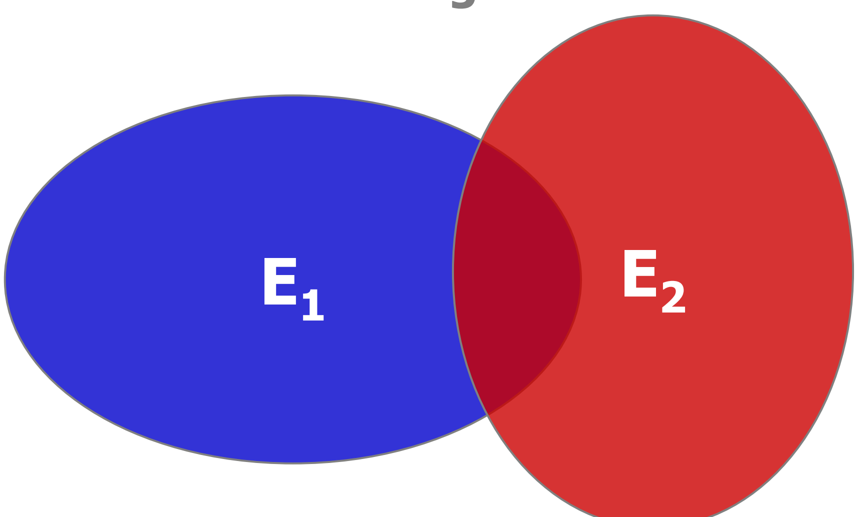
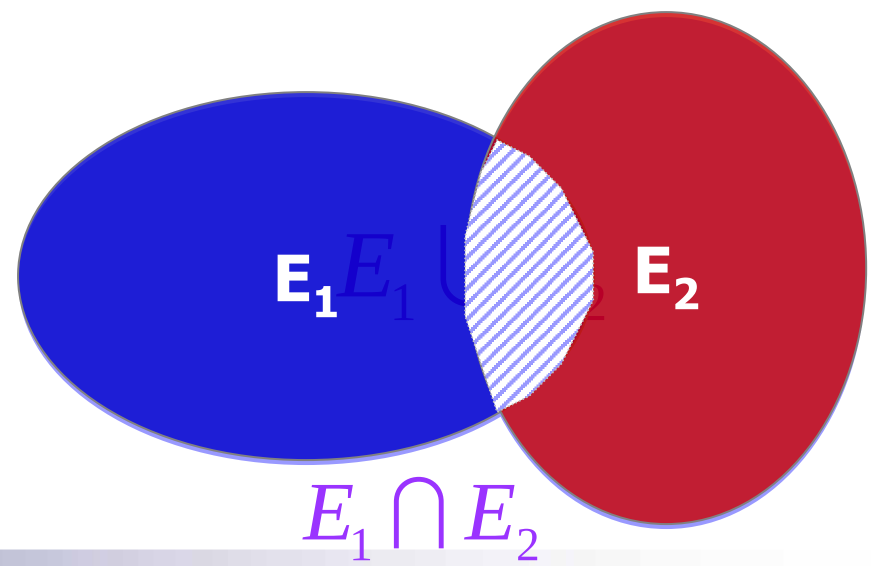




Conditional Probability

- **Probability of E_1 given that E_2 has happened**

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

$$P(E_1 \cap E_2) = P(E_1|E_2)P(E_2)$$ **; Multiplicative Law**

- **Independent Event**
  - The occurrence or nonoccurrence of one does not affect the probability of the other

$$P(E_1 \cap E_2) = P(E_1)P(E_2)$$

i.e. $$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = P(E_1)$$

Exercise

- **From the Penn State station data for Jan. 1980, compute the probability of precipitation, of T>32F, conditional probability of precipitation if T>32F, and conditional probability of precipitation tomorrow if it is raining today**

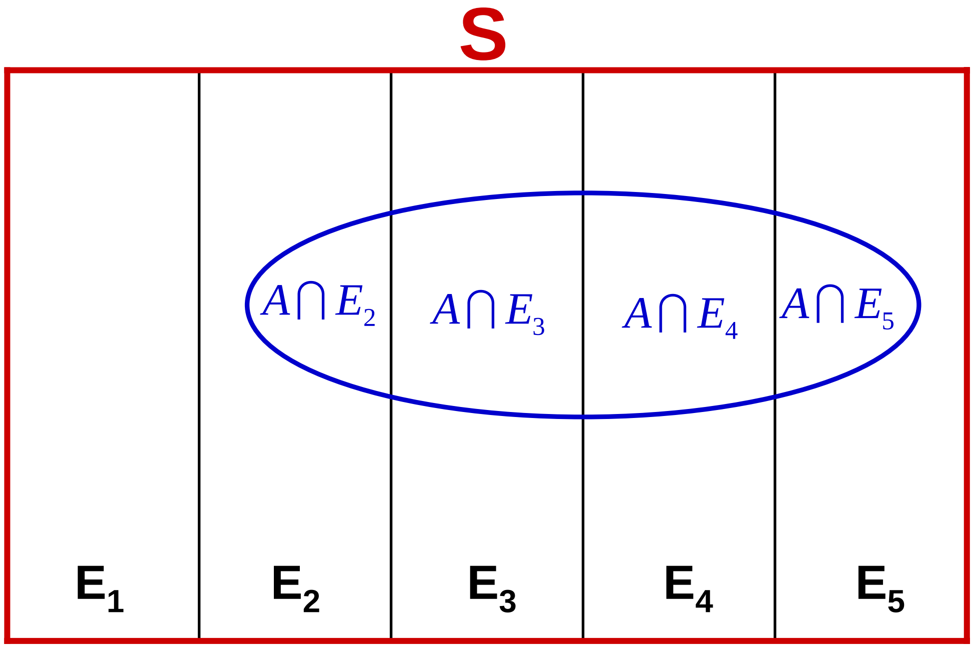
- **Prove graphically the DeMorgan Laws:**

$$P\left\{(A\cup B)^c\right\}=P\left\{A^c\cap B^c\right\};P\left\{(A\cap B)^c\right\}=P\left\{A^c\cup B^c\right\}$$

Total Probability

- **MECE events, $\{E_i\}$, $i=1, \ldots, I$**

$$P(A) = \sum_{i=1}^{I} P(A \cap E_i) = \sum_{i=1}^{I} P(A|E_i)P(E_i)$$



Bayes' Theorem

- **Bayes' theorem is used to "invert" conditional probabilities**
  - If $P(E_1|E_2)$ is known, Bayes' Theorem may be used to compute $P(E_2|E_1)$.

Multiplicative law

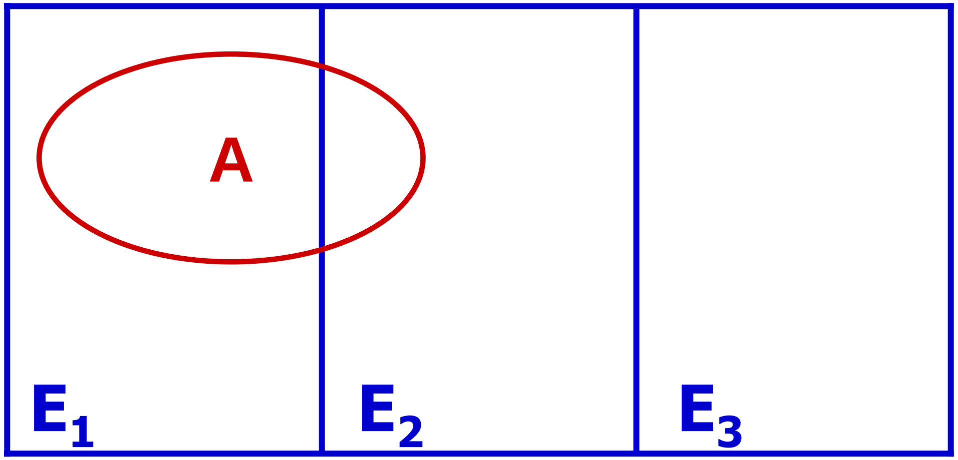
$$P(E_i|A) = \frac{P(E_i \cap A)}{P(A)} = \frac{P(A|E_i)P(E_i)}{P(A)} = \frac{P(A|E_i)P(E_i)}{\sum_{j=1}^{I} P(A|E_i)P(E_j)}$$

Law of total probability

  - Combines prior information with new information

Example of Bayesian Reasoning

- **Relationship between precipitation over SE US and El Nino**
  - Precipitation Events: E_1(above), E_2(normal), E_3(below) are **MECE**
  - El Nino Event: A
  - ***Prior information*** (from past statistics)
    - **$P(E_1)=P(E_2)=P(E_3)=33\%$**
    - **$P(A|E_1)=40\%$; $P(A|E_2)=20\%$; $P(A|E_3)=0\%$**



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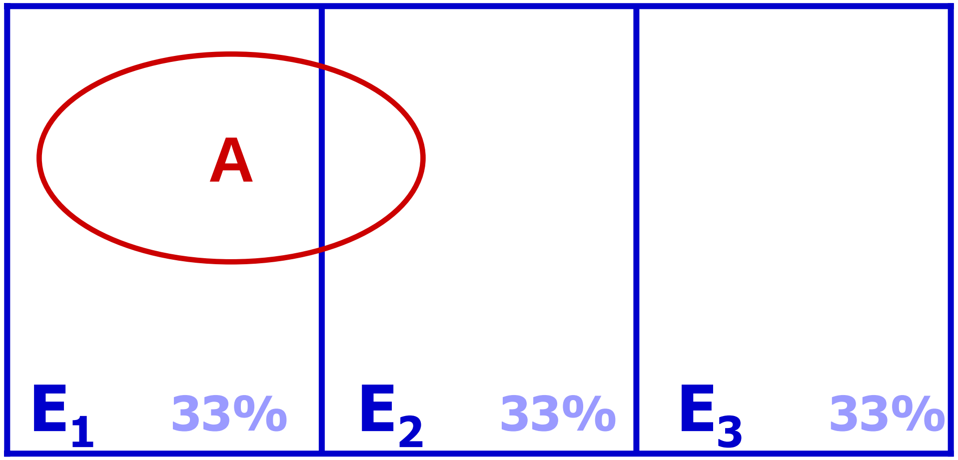

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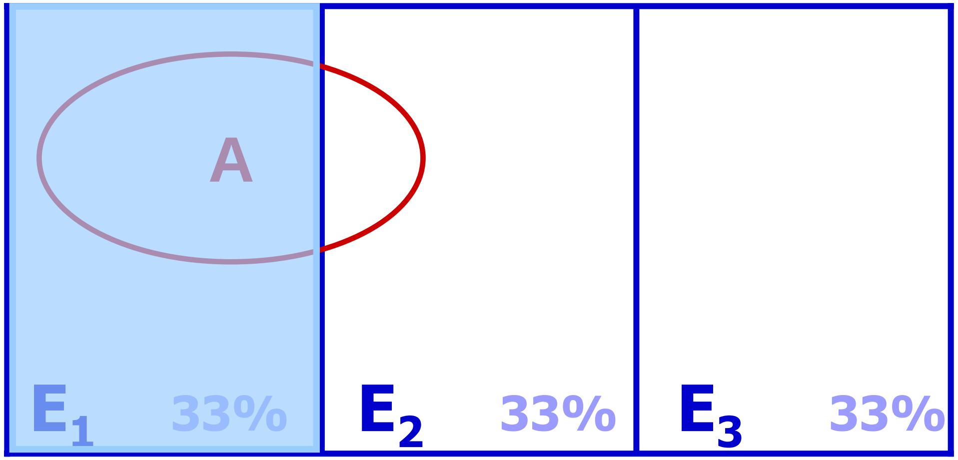

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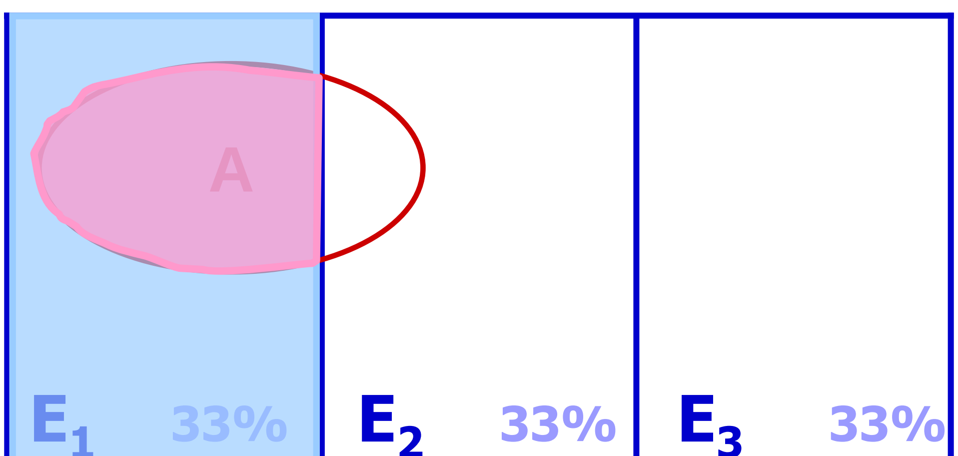
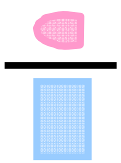


$$P(A|E_1)=\frac{P(A\cap E_1)}{P(E_1)}=$$



=0.40

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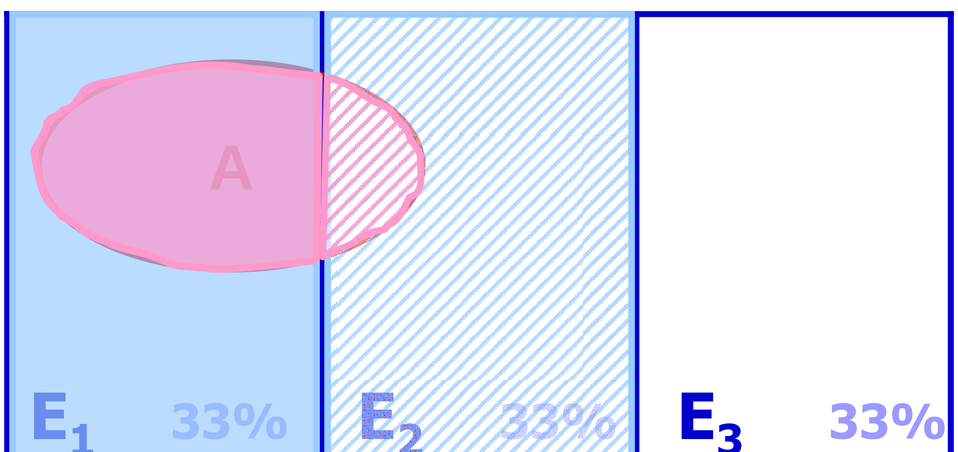
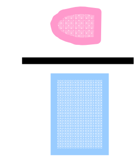
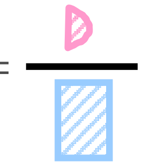


$$P(A|E_1)=\frac{P(A\cap E_1)}{P(E_1)}=$$  =0.40

$$P(A|E_2)=\frac{P(A\cap E_2)}{P(E_2)}=$$  =0.20

Example of Bayesian Reasoning

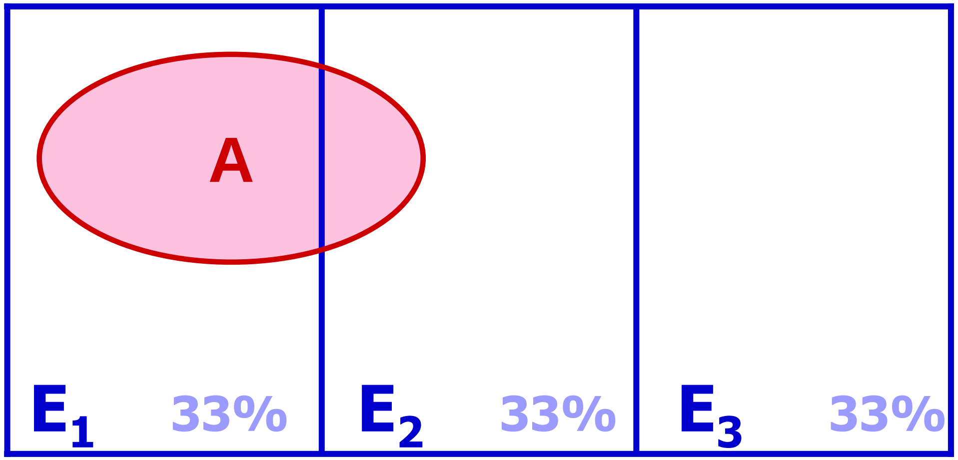
- **Total probability of A**

$$P(A) = \sum_{i=1}^{3} P(A | E_i) P(E_i)$$

$$= P(A | E_1) P(E_1) + P(A | E_2) P(E_2) + P(A | E_3) P(E_3)$$

$$= 0.4 * 0.33 + 0.2 * 0.33 + 0 * 0.33 = 0.20$$

- **NEW information: El Nino is happening! Probability of above normal precipitation?**



$$P(E_1 | A) = \frac{P(A \cap E_1)}{P(A)}$$

Example of Bayesian Reasoning

- **Total probability of A**

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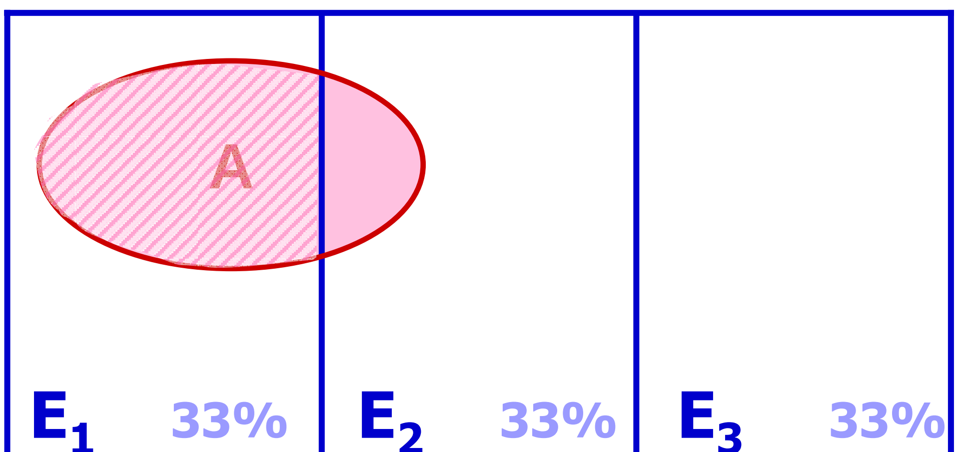
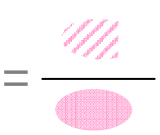


$$P(E_1|A) = \frac{P(A \cap E_1)}{P(A)}$$

$$= \frac{0.4 * 0.33}{0.2} = 0.66 =$$



Example of Bayesian Use in Variational Data Assimilation

- **Prior knowledge (measurement or forecast)**
  - T_1 of the true value T
- **New measurement, T_2**

$$P(T|T_2) = \frac{P(T_2|T)P_{prior,givenT_1}(T)}{P(T_2)} = \frac{\frac{1}{\sqrt{2\pi\sigma_1}} e^{-\frac{(T_2-T)^2}{2\sigma_2^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_2}} e^{-\frac{(T-T_1)^2}{2\sigma_1^2}}}{\frac{1}{\sqrt{2\pi\sigma_2}} e^{-\frac{(T_2-\bar{T})^2}{2\sigma_2^2}}}$$

$$\because P(T \cap T_2) = P(T_2|T)P(T)$$

Note The total probability of a measurement T_2 given a climatological average $\bar{T}$ is independent of T

Our best estimate of the true temperature *T*:
the value that maximizes (over *T*) the probability *P(T| T_2)*

$$\log P(T|T_2) = const - \frac{(T_2-T)^2}{2\sigma_2^2} - \frac{(T-T_1)^2}{2\sigma_1^2}$$

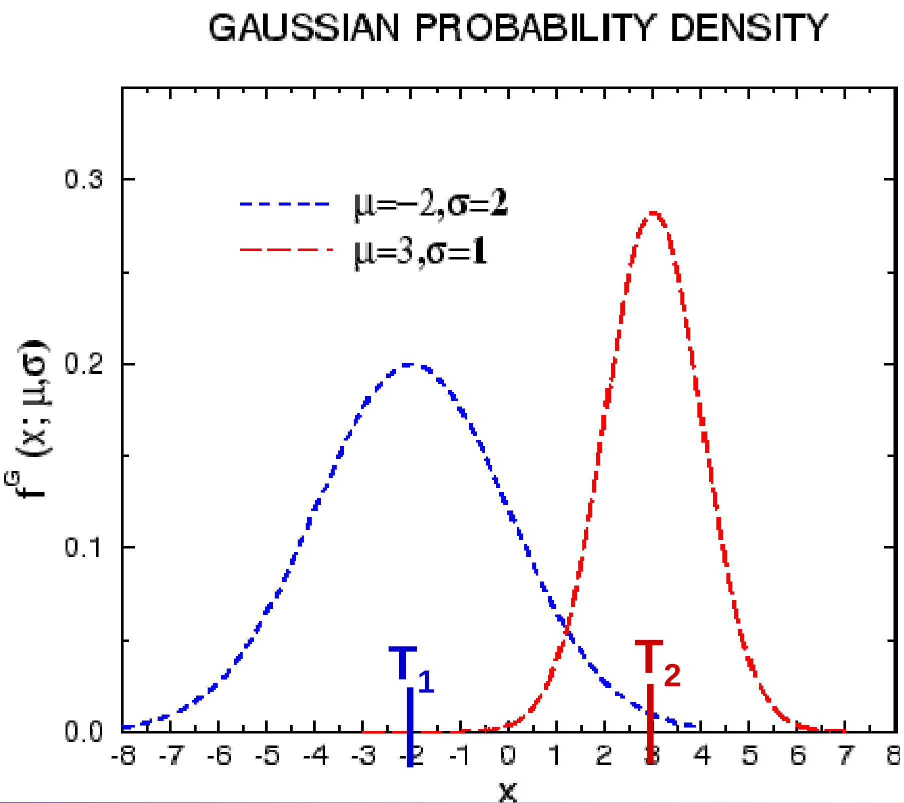
Or minimizes (over *T*) the cost function used in 3D-Var

$$J = \frac{(T_2-T)^2}{2\sigma_2^2} + \frac{(T-T_1)^2}{2\sigma_1^2}$$

Probability Density Function

- **Gaussian distribution with mean, *m_k*, & variance, *σ_k***

$$f_k(x) = \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp\left(-\frac{(x-m_k)^2}{2\sigma_k^2}\right)$$



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